Math 222A Lecture 10 Notes

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1 The Hopf-Lax Solution to Hamilton-Jacobi Equations

1.1 The Hamiltonian in classical mechanics

Last time, we were solving the Hamilton-Jacobi equation

$$\begin{cases} u_t + H(x, Du) = 0 \\ u(0) = u_0 \end{cases}$$

using the calculus of variations:

$$u(x,t) = \inf_{y(t)=x} \int_0^t L(y(s), \dot{y}(s))\, ds + u_0(y(0)).$$

Theorem 1.1. *The function u solves the Hamilton-Jacobi equation for as long as the solutions stay smooth.*

In the proof, we had the convex duality

$$H(x,p) = \max_q p \cdot q - L(x,q)$$

for the Hamiltonian $H(x,p)$ and the Lagrangian $L(x,q)$.

Example 1.1. Here is an example from classical mechanics. Consider the Lagrangian

$$L(x,q) = \frac{1}{2} m q^2 - \phi(x),$$

where $\frac{1}{2} m q^2$ is kinetic energy and $\phi(x)$ is potential energy. Then

$$H(x,p) = \sup_q p \cdot q - \frac{1}{2} m q^2 + \phi(x)$$

Complete the square to get

$$= \sup_q \frac{1}{2m}p^2 - \frac{1}{2m}(p-mq)^2 + \phi(x)$$
$$= \frac{1}{2m}p^2 + \phi(x)$$

In the physical interpretation, the Hamiltonian $H(x,p)$ plays the role of the energy of the system.

1.2 The Hopf-Lax formula

Now we will consider a special case, where $L = L(q)$ does not depend on x (and consequently $H = H(p)$). Assume that L, H are strictly convex and coercive (i.e. $\lim_{q\to\infty} \frac{L(q)}{|q|} = \infty$). The Euler-Lagrange equation tells us that

$$\cancel{L_x(y,\dot{y})} + \frac{d}{dt}L_q(y,\dot{y}) = 0.$$

So we get that $L_q(\dot{y})$ is constant. Since L_q is a local diffeomorphism, we get that $\dot{y}$ is constant. That is, the solutions to the Euler-Lagrange equation are linear.

We claim that fixing the endpoints $y(0), y(t)$, the minimum is attained for linear trajectories.

Theorem 1.2 (Hopf-Lax formula[1]). *If $L = L(q)$ is convex, then*

$$u(x,t) = \inf_y u_0(y) + tL\left(\frac{x-y}{t}\right).$$

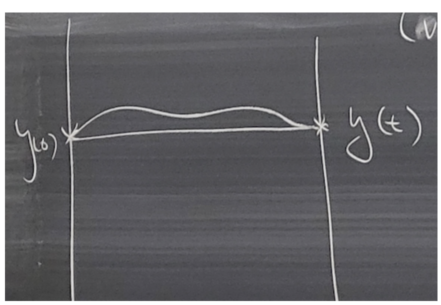
Proof. Since

$$\int_0^t \dot{y}(s)\,ds = y(t) - y(0),$$

we can average to get

$$\frac{1}{t}\int_0^t \dot{y}(s)\,ds = \frac{y(t)-y(0)}{t},$$

where the right hand side is the average velocity for a straight path.



[1]This is from the 50s or the 60s. Professor Tataru was actually able to meet Lax a few times.

Then

$$\int_0^t L(\dot{y}(s))\,ds = t \cdot \frac{1}{t}\int_0^t L(\dot{y}(s))\,ds$$

Convexity says that $L(\frac{x+y}{2}) \leq \frac{1}{2}(L(x)+L(y))$. More generally, we get that $L(hx+(1-h)y) \leq hL(x)+(1-h)L(y)$. If we use n variables, this is $L(\frac{x_1+\cdots+x_n}{n}) \leq \frac{1}{n}(L(x_1)+\cdots+L(x_n))$. If we increase the number of variables, this says that $L(\mathrm{avg}(z)) \leq \mathrm{avg}(L(z(s)))$, where we are taking average integrals. This is called **Jensen's inequality**, and it gives us

$$\geq t \cdot L\left(\frac{y(t)-y(0)}{t}\right)$$

In other words, the cost of an arbitrary path is $\geq$ the cost of the straight path. $\square$

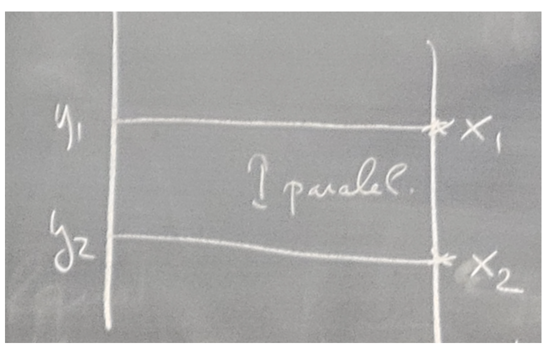
We are not done yet. We still need to minimize $u_0(y(0))$ over the choice of $y(0)$.

1.3 Properties of the Hopf-Lax solution

Assume L is convex and coercive. For simplicity, also assume that u_0 is bounded. Observe that if $t > 0$, then we can restrict $q = \frac{x-y}{t}$ to a compact set. So if u_0 is also continuous, then the infimum is attained.

Proposition 1.1. *If $u_0 \in \mathrm{Lip}$, then $u \in \mathrm{Lip}$.*

Proof. Here is a proof by picture. Suppose we have points x_1, x_2, and we want to compare $u(x_1)$ and $u(x_2)$. It is enough to consider parallel trajectories with y_1, y_2.



Take $x_1 - y_1 = x_2 - y_2$. Then $y_1 - y_2 = x_1 - x_2$. We have

$$u(x_1,t) = \inf_{y_1} u_0(y_1) + tL\left(\frac{x_1-y_1}{t}\right),$$

$$u(x_2,t) = \inf_{y_2} u_0(y_2) + tL\left(\frac{x_2-y_2}{t}\right).$$

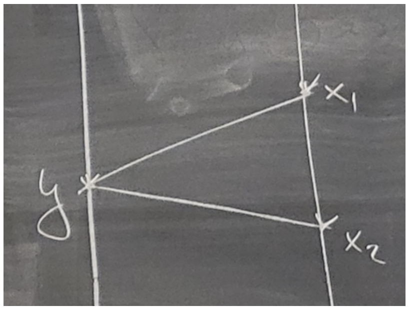
Using the Lipschitz condition, $|u_0(y_1) - u_0(y_2)| \le L|y_1 - y_2| = L|x_1 - x_2|$. So the conclusion is that

$$|u(x_1,t) - u(x_2,t)| \le L|x_1 - x_2|. \qquad \square$$

What if we don't assume u is Lipschitz? Can we still conclude that u is Lipschitz?

Proposition 1.2. *If u_0 is continuous, then $u(t)$ is Lipschitz.*

Proof. In this case, compare x_1 and x_2 to the same y:



We have

$$u(x_1) = \inf_y u_0(y) + L\left(\frac{x_1 - y}{t}\right),$$

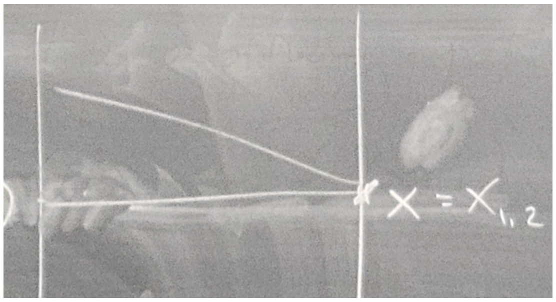
$$u(x_2) = \inf_y u_0(y) + L\left(\frac{x_2 - y}{t}\right).$$

The difference

$$\left| L\left(\frac{x_1 - y}{t}\right) - L\left(\frac{x_2 - y}{t}\right) \right| \le C \cdot \frac{|x_1 - x_2|}{t},$$

where the Lipschitz constant $C = C(t)$ in the set where $\frac{x_1 - y}{t}$ and $\frac{x_2 - y}{t}$ live.

Where should we look? $\frac{y - x_1}{t}$, $\frac{y - x_2}{t}$ cannot be too large. Let $x = x_1 = x_2$, and compare the straight trajectory to an arbitrary trajectory.

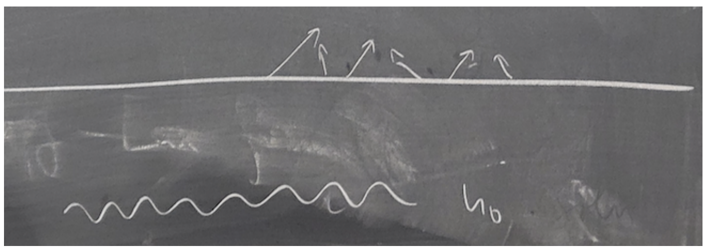


The oblique trajectory loses if $u_0(x)+tL(0) \leq u_0(y)+tL\left(\frac{x-y}{t}\right)$. This is when $\frac{2M}{t} \leq L(\frac{x-y}{t})$. So we can restrict to y such that $L(\frac{x-y}{t}) \leq \frac{2M}{t}$. So $\frac{x-y}{t}$ is in a compact set depending on t. Then the conclusion is that

$$|u(x_1,t) - u(x_2,t)| \leq C(t) \cdot \frac{|x_1 - x_2|}{t},$$

where $C(t)$ is the Lipschitz constant for L in the region $L(q) \leq \frac{C}{t}$. This Lipschitz constant goes to ∞ as $t \to 0$. □

In terms of the Hamilton-Jacobi equation, there will be lots of velocities with different speeds. So there is only an average velocity that survives.



We say that this PDE has a mild **regularizing effect**.

1.4 Almost everywhere solvability of the Hamilton-Jacobi equation

Recall the following theorem from real analysis (which requires measure theory).

Theorem 1.3. *If u is a Lipschitz function, then u is differentiable almost everywhere.*

So we get the following conclusion.

Corollary 1.1. *The solution u is differentiable almost everywhere.*

Proposition 1.3. *Let (x,t) be a differentiability point for u. Then the Hamilton-Jacobi equation holds at (x,t).*

Corollary 1.2. *The function u solves the Hamilton-Jacobi equation almost everywhere.*

Let's prove the proposition.

Proof. We can think of the Hamilton-Jacobi equation as proving two separate inequalities. If our trajectory is optimal, then it is optimal if we only look at the trajectory at a shorter length of time. Look at the optimal trajectory, ending at y and with slope $\frac{x-y}{t}$.

Then

$$u(x,t) = u_0(y) + tL\left(\frac{x-y}{t}\right),$$

so

$$u\left(x - h\frac{x-y}{t}, t-h\right) = u_0(y) + (t-h)L\left(\frac{x-y}{t}\right)$$

The first equation tells us that y is the optimal trajectory for (x,t), and the second says that y is optimal for $(x \cdot h\frac{x-y}{t}, t-h)$.

Let $q = \frac{x-y}{t}$. Then dividing by h gives

$$\frac{u(x,t) - u(x-hq, t-h)}{h} = hL(q).$$

Letting $h \to 0$ gives

$$\partial_x u \cdot q + \partial_t u = L(q).$$

So for this special q we have chosen,

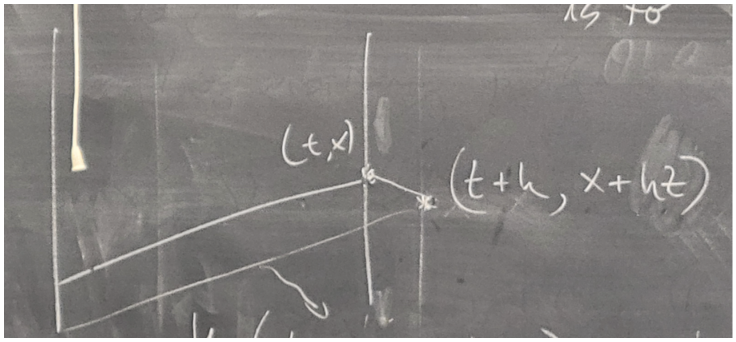
$$\partial_t u + \partial_x u \cdot q - L(q) = 0.$$

We want to think of this in terms of the Legendre transform. Since $H(p) = \sup p \cdot q - L(q)$, the latter half of our equation, $\partial_x u \cdot q - L(q)$, is $\leq H(\partial_x u)$. So we get

$$\partial_t u + H(\partial_x u) \geq 0.$$

Now we want to produce the other inequality. Notice that for the previous inequality, it was enough to work with a specific value of q, whereas for this direction, we will need to look at all values of q. Instead of looking at the past of (t,x), look at the future of (t,x).

Our picture looks like



One trajectory from $(t+h, x+hz)$ is to go through x, but this may not be optimal. So

$$u(t+h, x+hz) \leq u(t,x) + \underbrace{hL(z)}_{=\int_t^{t+h} L(z)\,ds}$$

As before, subtract the right hand side, divide by h, and let $h \to 0$. Then we get

$$\frac{u(t+h, x+hz) - u(t,x)}{h} \leq L(t) \implies \partial u + \partial_x uz \leq L(z).$$

So we have proven that for all z,

$$\partial_t u + \partial_x u \cdot z \leq 0.$$

Taking the supremum over all z, we get

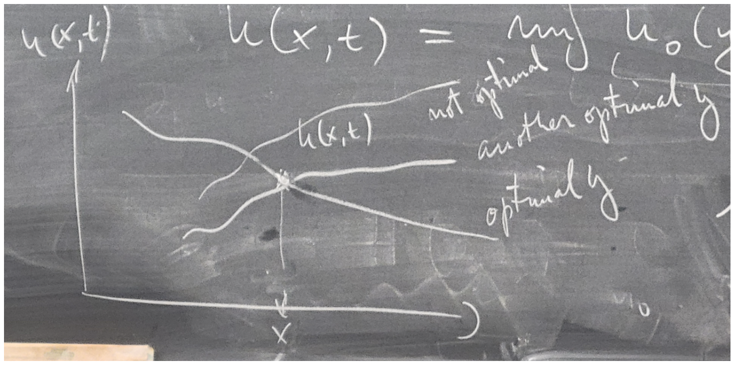
$$\partial_t u + H(\partial_x u) \leq 0. \qquad \square$$

Now we will tell a story. The details are in Evans' book, but the overall story is more important. We want to ask a question: Does solving the Hamilton-Jacobi equation almost everywhere suffice to guarantee uniqueness for Hamilton-Jacobi? Equivalently, does this guarantee that u is the minimal value function? The answer is no.

Are there other interesting properties for the function u? Look at the Hopf-Lax formula

$$u(x,t) = \inf u_0(y) + tL\left(\frac{x-y}{t}\right).$$

Observe that this is an infimum of functions which are smooth in x. We can compare what this looks like for different optimal/nonoptimal y:



Since we are taking a minimum, we can see that our curve could have a corner pointing upwards, but a corner pointing downwards is not possible. This points to a concavity property of our solution.

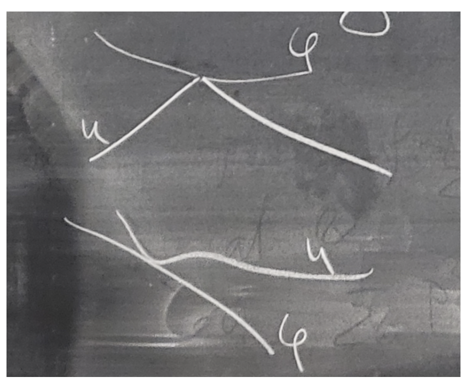
Proposition 1.4. *u is semiconcave.*

Concave means that $u(t,x) \geq \frac{u(t,x+y)+u(t,x-y)}{2}$. **Semiconcave** means that

$$u(t,x) \geq \frac{u(t,x+y)+u(t,x-y)}{2} - c \cdot |x-y|^2.$$

Theorem 1.4. *The optimal value function u is the unique semiconcave solution to the Hamilton-Jacobi equation.*

The proof is in Evans, but it is a little hard to follow. There is a better way to do things! Instead of plugging in u to check whether it satisfies the equation, if we have a corner, draw a tangent test function φ with $\varphi_t + H(\partial_x \phi) \geq 0$ or $\varphi_t + H(\partial_x \phi) \leq 0$.



These are called **viscosity solutions** for Hamilton-Jacobi equations.